# Odd triplet superconductivity in a superconductor/ferromagnet system with a spiral magnetic structure

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We analyze a superconductor-ferromagnet ($S/F$) system with a spiral magnetic structure in the ferromagnet F for a weak and strong exchange field. The long-range triplet component (LRTC) penetrating into the ferromagnet over a long distance is calculated for both cases. In the dirty limit (or weak ferromagnetism) we study the LRTC for conical ferromagnets. Its spatial dependence undergoes a qualitative change as a function of the cone angle $\vartheta$. At small angles $\vartheta$ the LRTC decays in the ferromagnet exponentially in a monotonic way. If the angle $\vartheta$ exceeds a certain value, the exponential decay of the LRTC is accompanied by oscillations with a period that depends on $\vartheta$. This oscillatory behavior leads to a similar dependence of the Josephson critical current in $SFS$ junctions on the thickness of the $F$ layer. In the case of a strong ferromagnet the LRTC decays over the length which is determined by the wave vector of the magnetic spiral and by the exchange field.

## I. INTRODUCTION

It is well known that superconductivity and ferromagnetism are antagonistic phenomena (see, for example, the reviews[1–3]). Exchange interaction in ferromagnets results in ordering electron spins in one direction, whereas superconducting correlations in conventional superconductors lead to the formation of Cooper pairs with opposite spins of electrons. The antagonistic character of ordering in ferromagnets ($F$) and in superconductors ($S$) is the reason for an essential difference between the proximity effects in $S/N$ and $S/F$ structures (here $N$ denotes a normal nonmagnetic metal). In $S/N$ structures the condensate penetrates the normal metal $N$ over a rather long distance which in the dirty limit ($\tau T \ll 1$, $\tau$ is the elastic scattering time) is equal to

$$\xi_N = \sqrt{D/2\pi T}, \tag{1}$$

where $D = vl/3$ is the diffusion coefficient and $l = v\tau$ is the mean free path. On the other hand, the depth of the condensate penetration into a ferromagnet in $S/F$ system is much shorter

$$\xi_F = \sqrt{D/h} \tag{2}$$

if the exchange energy $h$ is larger than the temperature $T$ (usually $h \gg T$). The formula (2) is valid for a short mean free path ($h\tau \ll 1$). If the exchange field $h$ is strong enough ($h\tau \gg 1$), the condensate penetrates the ferromagnet over a distance of the order of $l$ (if $\tau T < 1$) and oscillates with the period $\sim v/h$.[4,5]

Note that a short length of the condensate penetration is related to the fact that Cooper pairs in a conventional superconductor are formed by two electrons with opposite spins. In the case of a homogeneous magnetization the wave function $f(t-t')$ of Cooper pairs penetrating into the ferromagnet consists of two parts

$$f_3(t-t') \sim \langle \psi_\uparrow(t)\psi_\downarrow(t') - \psi_\downarrow(t')\psi_\uparrow(t) \rangle, \tag{3}$$

$$f_0(t-t') \sim \langle \psi_\uparrow(t)\psi_\downarrow(t') + \psi_\downarrow(t')\psi_\uparrow(t) \rangle. \tag{4}$$

The first function describes the singlet component. It differs from zero both in the whole superconducting region and in the ferromagnet over the length $\xi_F$. The second function $f_0(t-t')$ describes the triplet component with zero projection of the magnetic moment of a Cooper pair on the $z$ axis (the magnetization vector **M** is oriented along the $z$ axis). It is not zero only in the vicinity of the $S/F$ interface, over a distance $\sim \xi_S = \sqrt{D/2\pi T_c}$ in the superconductor and over the distance $\xi_F$ in the ferromagnet. This function is an odd function of the difference $(t-t')$ and therefore is equal to zero at $t=t'$. This means that in the Matsubara representation $f_0(\omega)$ is an odd function of $\omega$, whereas the function $f_3(\omega)$ is an even function of $\omega$.

All the statements above concern only the case of a homogeneous magnetization in the $F$ region. The situation changes qualitatively if the magnetization is not homogeneous, for example, if one has a spiral magnetic structure in the ferromagnet. In this case not only the singlet and triplet component with zero projection of magnetic moment, but also a triplet component with a nonzero projection of magnetic moment arises in the system. This type of the triplet component means that Cooper pairs appear in the system that are described by the condensate function $f_{tr}(t-t') \sim \langle \psi_\uparrow(t)\psi_\uparrow(t') \rangle$ or $f_{tr}(t-t') \sim \langle \psi_\downarrow(t)\psi_\downarrow(t') \rangle$. The triplet pairing is well known in superfluid He[3] (Refs. 6 and 7) or in $Sr_2RuO_3$[8] and is believed to be realized in compounds with heavy fermions.[9] The condensate function $f_{tr}(\omega)$ in these materials is an odd function of momentum **p** and an even function of the Matsubara frequency $\omega$. The order parameter $\Delta$ is a sum of $f_{tr}(\omega)$ over positive and negative $\omega$ and corresponds to a triplet pairing. This triplet component is suppressed by impurity scattering.[10]

In $S/F$ structures the impurity scattering is rather strong as ferromagnetic films used in these structures are thin (typically the thickness of the $F$ films $d$ is about 20–100 Å) and

the elastic scattering at least at the $F$ surface is strong. Therefore the conventional triplet component would be strongly suppressed. However there is a special type of the triplet component which can survive a strong impurity scattering. This triplet component is described by a condensate wave function $f_{tr}(\omega)$ that is even in momentum **p** and odd in frequency $\omega$. This type of the condensate was suggested by Berezinskii in 1975 in attempt to describe the pairing mechanism in superfluid He$^3$.[11] It turned out however that in reality the condensate function in He$^3$ is an even function of $\omega$ and an odd function of **p**. Later a possibility to realize the odd (in frequency) triplet superconductivity in solids was discussed for various models in Refs. 12–14.

The odd in frequency $\omega$ and even in momentum **p** triplet component in $S/F$ structures differs from that discussed in the preceding paragraph. It coexists with the singlet component and the order parameter $\Delta$ is determined only by the ordinary (BCS) singlet component $f_{sngl}(\omega)$ even in $\omega$. The triplet component with nonzero projection of the magnetic moment of Cooper pairs arises as the result of action of a rotating exchange field on electron spins. This type of the condensate penetrates the ferromagnet over a long distance of the order of $\xi_N$ [see Eq. (1)] provided the period of the magnetization rotation exceeds $\xi_N$.

A $S/F$ structure with an inhomogeneous magnetization has been studied for the first time in Ref. 15. The authors considered a $S/F$ structure with a Bloch-type domain wall at the $S/F$ interface in the limit of a short mean free path ($h\tau < 1$). In the domain wall of the thickness $w$ the magnetization vector was supposed to have the form **M** $= M_0\{0, \sin Qx, \cos Qx\}$, where the $x$ axis is normal to the $S/F$ interface. Outside the domain wall the magnetization was constant: $M = M_0\{0, \sin Qw, \cos Qw\}$. The condensate function $f_{tr}(\omega, x) \sim \langle \psi_\alpha(t)\psi_\alpha(t') \rangle_\omega$ was found from the linearized Usadel equation. It was established that this triplet component odd in frequency and even in momentum penetrates the magnetic domain wall over the length

$$\xi_Q = [Q^2 + 2|\omega|/D]^{-1/2}. \qquad (5)$$

It spreads outside the domain wall over distances of the order $\xi_N$. At $x > w$ the vector of the magnetization is fixed so that the first term in Eq. (4) $Q^2$ should be dropped. This triplet component may be called the long-range triplet component (LRTC). The LRTC may cause a significant change in the conductance of the Andreev interferometer consisting of ferromagnetic wires and a superconducting loop. As was shown in Ref. 15, the conductance variation decreases with increasing temperature in a monotonic way.

Somewhat later the same problem was considered in the Ref. 16. In that paper, a more complicated situation was discussed, namely, the case when the width of the domain wall is short compared to the mean free path. In order to find the condensate function (quasiclassical Green's function) $f_{tr}(\omega, x)$ in this case, one needs to solve a more general Eilenberger equation taking into account a nonhomogeneous magnetization. Unfortunately, the authors of Ref. 16 did not manage to solve the Eilenberger equation and therefore restricted themselves with a rough estimation of the amplitude of
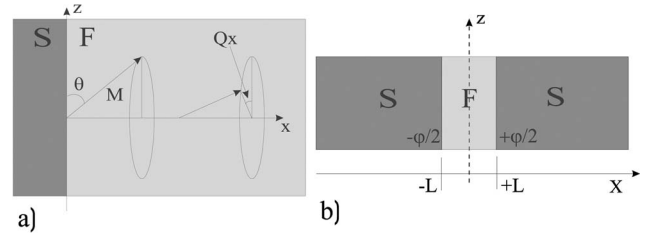


FIG. 1. Schematic view of the system under consideration. (a) $S/F$ structure with a conical ferromagnet. Magnetization in the ferromagnet F rotates in the $\{y, z\}$ plane and has a constant projection on the $x$ axis. (b) Josephson junction with a conical ferromagnet.

$f_{tr}(\omega, x)$. So, the problem of calculation of the odd triplet condensate function in the ballistic regime remained unsolved.

In the present paper we continue studying behavior of the triplet condensate considering new situations. To be specific, we study the LRTC in $S/F$ junctions with a spiral magnetic structure in different limits including the quasi-ballistic one (i.e., $h\tau > 1$). This spiral structure may be both an intrinsic property of a ferromagnet (for example, a helicoidal structure; see Ref. 17) or may just serve as a rough model for magnetic domains.

We consider a system with a conical ferromagnet, that is, we assume that the magnetization in F rotates in $\{y, z\}$ plane and has a constant component along the $x$ axis. This type of spin structures is realized, for instance, in Ho (Ref. 18) and what we discuss now is a generalization of the problem considered in Ref. 15, where the $x$ component of the magnetization in F was assumed to be zero. Our study of the proximity effect in such a $S/F$ system with a spiral magnetic structure is motivated also by the recent experiment performed on a Al/Ho structure.[19]

First, solving the Usadel equation, we find the condensate function in the dirty limit. It turns out that the spatial variation of the LRTC has a nontrivial dependence on the cone angle $\vartheta$ (see Fig. 1). If $\vartheta$ is small, the LRTC decays with $x$ exponentially over a distance of the order of $\xi_Q$, but at angles $\vartheta > \sin^{-1}(1/3) \approx 19°$, the LRTC decays in a nonmonotonic way. It oscillates with a period depending on $\vartheta$. These oscillations lead to oscillations of the critical Josephson current in $SFS$ junction as a function of the thickness of the $F$ film $2L$ if the thickness $2L$ is essentially greater than $\xi_F$. In this case the Josephson coupling is only due to the LRTC.

We continue the investigation studying in the third section the LRTC in a $S/F$ structure with a spiral structure in the limit when the condition $h\tau \gg 1$ is valid and the Eilenberger equation must be solved. We analyze peculiarities of the LRTC in this case. Surprisingly, this case has not been investigated previously, although it may correspond to a real situation. In many ferromagnets the exchange energy $h$ is very large so that the product $h\tau$ can be arbitrary. In both sections we make an assumption that the proximity effect is weak, i.e., the amplitude of the condensate function in the ferromagnet $f$ is small. This assumption is presumably valid in most cases because there is a strong reflection of electrons at the $S/F$ interface due to a considerable mismatch in electronic parameters between the ferromagnet and

superconductor.[20] In conclusion we summarize the obtained results and discuss a possibility of experimental observations of the predicted behavior of the LRTC.

Note that in Refs. 15 and 16 and in the present paper a Bloch-type spiral structure is analyzed. The rotation axis was assumed to be perpendicular to the $S/F$ interface so that the condensate decays in the ferromagnet in the direction parallel to this axis (see Fig. 1). In this case the LRTC arises in the system.

At the same time, one can imagine another, Neel-like type of a spiral structure with the magnetization vector $M$ that rotates in the plane of the ferromagnetic film and does not vary in the perpendicular direction. A solution for this type of the spiral structure has been found in Ref. 21. The authors considered the case of a thin F film and did not study the decay length in the direction normal to the film.

The same problem for a thick (infinite) ferromagnetic film was analyzed by Champel and Eschrig in a recent paper.[22] These authors assumed that the magnetization vector $M$ lies in the plane of the ferromagnetic film and rotates in this plane being constant in the perpendicular direction. Surprisingly, they found that there was no LRTC in this case.

As was shown later,[23] the absence of LRTC is specific for this homogeneous spiral magnetic structure. In contrast, the long-range triplet component arises if the Neel-type spiral structure is nonhomogeneous; for example, if the $F$ film consists of magnetic domains separated by the Neel walls. In this case the LRTC appears in domain walls and decays in domains over a long length of the order $\xi_N$. Another case where the LRTC arises is a spin-active $S/F$ interface. Such a type of interface leads to mixing singlet and triplet components and the triplet component may penetrate even into a half-metal ferromagnet when the conduction band consists of electrons with only one direction of spins.[24]

## II. S/F STRUCTURE WITH A ROTATING MAGNETIZATION. DIRTY LIMIT

We consider a system shown schematically in Fig. 1. A ferromagnetic film $F$ is attached to a superconductor $S$. The magnetization $M$ in the ferromagnet is assumed to rotate in space and the vector $\mathbf{M}$ has the form

$$\mathbf{M} = M_0\{\sin \vartheta, \cos \vartheta \sin Qx, \cos \vartheta \cos Qx\}. \quad (6)$$

This form of the magnetization vector implies that the projection of the $\mathbf{M}$ vector onto the $x$ axis $M_0\sin \vartheta$ is constant and the projection onto the $\{y,z\}$ plane $M_0\cos \vartheta$ rotates in space with the wave vector $Q$. We also assume that the condition

$$h\tau \ll 1 \quad (7)$$

is fulfilled, that is, either the exchange energy $h$ is not large or the collision frequency $\tau^{-1}$ is high enough. In addition, we assume that the proximity effect is weak, i.e., the condensate function $f$ is small. The smallness of the $f$ function means the presence of a barrier at the $S/F$ interface or a big mismatch in electronic parameters of the superconductor and ferromagnet (the Fermi momenta in $F$ and $S$ differ greatly). In this case one can linearize the Usadel equation and represent it in the form[4,25]

$$\partial^2 \check{f}/x^2 - 2k_\omega^2 \check{f} + ik_h^2 \mathrm{sgn}\, \omega\{\cos \vartheta[[\hat{\sigma}_3,\check{f}]_+\cos \alpha(x)$$
$$+ \hat{\tau}_3[\hat{\sigma}_2,\check{f}]_+\sin \alpha(x)\} + \sin \vartheta \hat{\tau}_3[\hat{\sigma}_1,\check{f}]\} = 0, \quad (8)$$

where $\omega = \pi T(2n+1)$, $k_\omega^2 = |\omega|/D$, $k_h^2 = h/D$, the brackets $[\hat{\sigma}_3,\check{f}]_+$ and $[\hat{\sigma}_3,\check{f}]$ denote anticommutator and commutator. The quasiclassical condensate Green's function $\check{f}$ is a $4\times4$ matrix in the particle-hole and spin space. The Pauli matrices $\hat{\tau}_i$ and $\hat{\sigma}_i$ operate in the particle-hole and spin space, respectively.

The Green's function in the bulk of the superconductor is

$$\check{f}_S = \hat{\tau}_2\hat{\sigma}_3 f_S, \quad (9)$$

with $f_S = \Delta/\sqrt{\omega^2+\Delta^2}$. This function describes an ordinary singlet condensate in conventional superconductors. Assuming this solution to be valid up to the interface, we use the "rigid" boundary condition at the $SF$ interface $(x=0)$[26,27]

$$\partial \check{f}/\partial x = -\check{f}_S/\gamma_B, \quad (10)$$

where $\gamma_B = R_B\sigma$, $\sigma$ is the conductivity of the ferromagnet, and $R_B$ is the interface resistance per unit area. This boundary condition is valid provided the ratio $\xi_h/\gamma_B = \rho\xi_h/R_B$ is large, that is, the interface resistance should be larger than the resistance of the F wire of length $\xi_h$.

What we have to do is solve Eq. (8) in the ferromagnetic region $(x>0)$ with the boundary conditions (10) at $x=0$. Equation (8) is a linear differential equation with coefficients depending on the coordinate $x$. This dependence can be excluded if we make an unitary transformation determined by a matrix $\check{U}$ and introduce a new matrix $\check{f}_n$

$$\check{f} = \check{U}\check{f}_n\check{U}^+, \quad (11)$$

where the matrix $\check{U} = \exp(i\hat{\tau}_3\hat{\sigma}_1 Qx/2)$ describes a rotation in the spin and particle-hole space. Substituting the expression (11) into Eq. (8), we get the equation for the new matrix $\check{f}_n$

$$-\partial^2 \check{f}/\partial x^2 + (Q^2/2 + 2k_\omega^2)\check{f} + (Q^2/2)(\hat{\sigma}_1\check{f}\hat{\sigma}_1) - iQ\hat{\tau}_3[\hat{\sigma}_1,\partial\check{f}/\partial x]_+$$
$$- ik_h^2\mathrm{sgn}\, \omega\{\cos \vartheta[\hat{\sigma}_3,\check{f}]_+ + \sin \vartheta \hat{\tau}_3[\hat{\sigma}_1,\check{f}]\} = 0. \quad (12)$$

For brevity we dropped the subindex $n$. The boundary condition (10) acquires the form

$$\partial \check{f}/\partial x + i(Q/2)\hat{\tau}_3[\hat{\sigma}_1,\check{f}]_+ = -\check{f}_S/\gamma_B. \quad (13)$$

As concerns the matrix $\check{f}_S$, it does not change [it is invariant with respect to the transformation (11)]. In order to solve Eq. (12) with the boundary condition (13), we represent the matrix $\check{f}(x)$ as an expansion in Pauli matrices

$$\check{f}(x) = \sum_i \hat{\sigma}_i\hat{F}_i(x), \quad (14)$$

where $i=0,1,2,3$ and $\hat{\sigma}_0$ is the unit matrix. The functions $\hat{F}_i(x)$ are matrices in the particle-hole space. The spatial dependence of the functions $\hat{F}_i(x)$ is determined by the exponential functions $\hat{F}_i(x) \sim A_i\exp(-\kappa x)$, where the eigenvalues $\kappa$ are determined from the determinant of Eq. (12). Putting
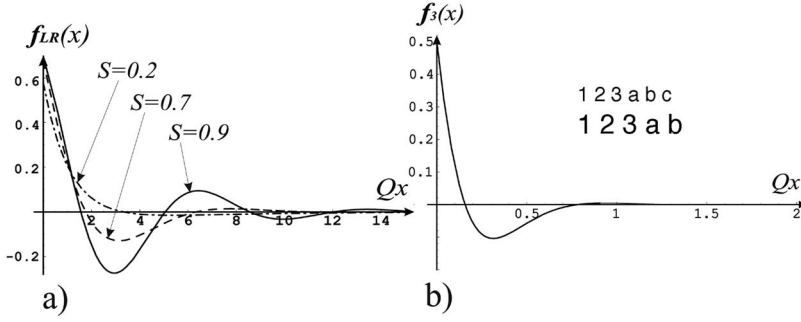
FIG. 2. Spatial dependence of the real part of the triplet (a) and singlet (b) component for different angles; $S = \sin \vartheta$. The singlet component almost does not depend on $\vartheta$. Both component are normalized to the quantity $(f_S / \gamma \kappa_h)$. The parameter $\kappa_h / Q$ is choosen equal to 5.

the determinant to zero, we obtain the equation for eigenvalues $\kappa$ (see Appendix A)

$$(\kappa^2 - Q^2 - 2\kappa_\omega^2)^2 [(\kappa^2 - 2\kappa_\omega^2)^2 + 4\kappa_h^4 \sin^2 \vartheta] + 4(Q\kappa)^2$$
$$\times [(\kappa^2 - 2\kappa_\omega^2)^2 + 4\kappa_h^4 \sin^2 \vartheta] + 4\kappa_h^4 \cos^2 \vartheta (\kappa^2 - Q^2 - 2\kappa_\omega^2)$$
$$\times (\kappa^2 - 2\kappa_\omega^2) = 0. \tag{15}$$

This equation has four pairs of roots. However, four of them correspond to solutions growing in the ferromagnet. As we are interested only in decaying solutions (Re $\kappa > 0$), we keep four proper roots.

In order to simplify the calculations, we consider a limiting case of a large exchange field assuming that

$$\kappa_h^2 \gg \{Q^2, \kappa_\omega^2\}. \tag{16}$$

In this limit two eigenvalues of the wave vectors $\kappa$ are large and equal to

$$\kappa_\pm = (1 \pm i \, \text{sgn} \, \omega) \kappa_h. \tag{17}$$

These roots describe a rapid decay of the condensate in the ferromagnet over an "exchange length" $\kappa_h^{-1}$ of the order $\xi_F$ [see Eq. (2)]. Only these eigenvalues appear in the case of a homogeneous magnetization. These large eigenvalues have both real and imaginary parts and they are equal to each other. This means that corresponding eigenfunctions decay and oscillate in space on the same scale. The oscillations of the function $F_3$ (singlet component) lead to oscillations of the critical temperature $T_c$ and Josephson current $I_c$ in $SFS$ structures with varying the $F$ film thickness $d$.[2,3,25]

In addition to these large values $\kappa_\pm$, there are two other solutions $\kappa_{a,b}$ that describe a long-range penetration of the triplet component. The singlet component also contains a part decaying slowly but, as we will see, its amplitude is small in comparison with the amplitude of the triplet component. These eigenvalues are small: $\kappa_{a,b}^2 \approx \max\{Q^2, \kappa_\omega^2\}$. One can find exact expressions for $\kappa_{a,b}$, but in order to make results more transparent, we represent solutions in the limit

$$Q^2 \gg \kappa_\omega^2. \tag{18}$$

In this case Eq. (15) is reduced to the following quadratic equation for the eigenvalues $\zeta_{a,b} \equiv \kappa_{a,b}^2 / Q^2$:

$$\zeta^2 - \zeta(1 - 3\sin^2 \vartheta) + \sin^2 \vartheta = 0 \tag{19}$$

with the roots

$$\zeta_{a,b} = 3/2[1/3 - \sin^2 \vartheta \pm \cos \vartheta \sqrt{1/9 - \sin^2 \vartheta}]. \tag{20}$$

Now, let us consider these eigenvalues as a function of the angle $\vartheta$. If the magnetization in the ferromagnet lies in $\{y, z\}$ plane (i.e., $\vartheta = 0$), we obtain only one root: $\zeta_a = 1$ (the other root should be dropped as it corresponds to a solution with the zero amplitude). This means that the triplet component penetrates the ferromagnet over the long-range (LR) distance $\xi_{LR} \sim Q^{-1}$. If the value of $Q^2$ is comparable with or less than $\kappa_\omega^2$, then the characteristic penetration length of the triplet component is $\xi_{LR} = 1/\sqrt{Q^2 + \kappa_\omega^2}$ (compare with Ref. 15). As follows from Eq. (20), at small $\vartheta$ the first eigenvalue equals $\zeta_a \simeq 1 - 4\sin^2 \vartheta \approx 1 - 4\vartheta^2$, whereas the second root is small and equal to $\zeta_b \simeq \sin^2 \vartheta \approx \vartheta^2$. Therefore in this case the exponential decay of the triplet component is slow: $f_{tr} \sim \exp(-Qx|\vartheta|)$.

Both the eigenfunctions decay exponentially in a monotonic way over a length much longer than $\xi_F$. The situation changes if $\sin \vartheta$ exceeds the value $1/3$ [$\theta \gtrsim \sin^{-1}(1/3)$]. In this case the LRTC oscillates and decays in a nonmonotonic way. The period of the oscillations $\sim 1/Q$ is much longer than $\xi_F$. If the magnetization vector is oriented almost along the $x$ axis ($\sin \vartheta \to 1, \cos \vartheta \to 0$), the period of oscillations $\sim Q^{-1}$ is much shorter than the decay length of the LRTC which is equal to $\sqrt{2}/(Q \cos \vartheta) \gg 1/Q$ (see Fig. 2).

The amplitudes of the components $\hat{F}(x)_i$ are found from the boundary conditions (13). One can show that the matrices $\hat{F}(x)_i$ can be represented in the form (see Appendix A)

$$\hat{F}(x)_{0,3} = \hat{\tau}_2 \sum_k A_{0,3k} \exp(-\kappa_k x), \tag{21}$$

$$\hat{\tau}_3 \hat{F}(x)_{1,2} = \hat{\tau}_2 \sum_k A_{1,2k} \exp(-\kappa_k x), \tag{22}$$

where the summation is carried out over all eigenvalues: $k = \pm; a, b$. In the approximation Eq. (16) the amplitudes are

$$\mp A_{0\pm}/\cos \vartheta = \pm iA_{2\pm}/\sin \vartheta = A_{3\pm} = (f_S/2\gamma_B \kappa_\pm), \tag{23}$$

$$-i \tan \vartheta A_{2a,b} = A_{0a,b} = (\zeta_{a,b} - 1)/(2i\sqrt{\zeta_{a,b}})A_{1a,b}. \tag{24}$$

The amplitude of the long-range triplet component is equal to

$$A_{1a} = \text{sgn}\,\omega \frac{f_S}{\gamma_B \kappa_h} \frac{\zeta_a \cos\vartheta}{(\zeta_a + 1)(\sqrt{\zeta_a} - \sqrt{\zeta_b})}. \tag{25}$$

The expression for $A_{1b}$ is obtained from Eq. (25) by permutation $a \rightleftarrows b$. These formulas are valid if both $\sin\vartheta$ and $\cos\vartheta$ are not too small: $\{\cos\vartheta, \sin\vartheta\} \gg Q/\kappa_h$.

As we mentioned, the amplitude $A_3$ corresponds to the singlet component and the amplitude $A_0$ describes the triplet component with zero projection of the magnetic moment on the $z$ axis. In a homogeneous case ($M$ is constant in the ferromagnet) this component penetrates the ferromagnet over a short length of the order $\xi_F$. In the case of rotating magnetization it penetrates the ferromagnetic region over a long distance of the order of $\min\{1/Q, 1/\kappa_\omega\}$. The amplitude $A_2$ arises only in the case of a tilted magnetization ($\vartheta \neq 0$). One can see from Eqs. (23)–(25) that in the considered limits the amplitudes $A_{3\pm}, A_{0\pm}, A_{2\pm}, A_{0a,b}, A_{2a,b}$, and $A_{1a,b}$ are comparable with each other (other amplitudes are small). Therefore the LRTC with nonzero projection on the $z$ axis is comparable with the magnitude of the singlet component at the $S/F$ interface $(A_{3+} + A_{3-}) \sim f_S/\gamma \kappa_h$. The spatial dependence of the LRTC amplitude is given by the expression

$$f_{LR}(x) = \frac{f_S \text{sgn}\,\omega}{\gamma_B \kappa_h}\left[\frac{\zeta_a}{\zeta_a + 1}\exp(-Qx\sqrt{\zeta_a}) - \frac{\zeta_b}{\zeta_b + 1}\right.$$
$$\left. \times \exp(-Qx\sqrt{\zeta_b})\right]\frac{\cos\vartheta}{\sqrt{\zeta_a} - \sqrt{\zeta_b}}, \tag{26}$$

where $f_{LR}(x)$ is defined in this way: $\hat{F}_1(x) = i\hat{\tau}_1 f_{LR}(x), x \gg \xi_F$.

As it should be, this function is an odd function of $\omega$. In Fig. 2 we plot the spatial dependence of $\text{Re}[f_{LR}(x)]$ for some values of $\sin\vartheta$ and compare it with the spatial variation of the singlet component.

$$f_3(x) = \frac{f_S}{2\gamma_B}\left[\frac{1}{\kappa_+}\exp(-\kappa_+ x) + \frac{1}{\kappa_-}\exp(-\kappa_- x)\right] \tag{27}$$

It is seen that the LRTC decays over distances longer than the singlet component. At small $\sin\vartheta$ the LRTC decays monotonously, but with increasing $\vartheta$, oscillations of the LRTC arise. The characteristic scale of the LRTC decay in the considered case [the condition (18) is fulfilled] is $Q^{-1}$. In the opposite limit of a spiral with a small $Q$ ($Q \ll \kappa_\omega$) the roots $\kappa_{a,b}$ changes: $\kappa_{a,b} \approx \sqrt{2}\kappa_\omega(1 \pm iQ\sin\vartheta/\sqrt{2}\kappa_\omega)$. This means that the LRTC decreases exponentially in the ferromagnet on the length of the order $\xi_N$ and oscillates with the period $(Q\sin\vartheta)^{-1}$. In this case the decay length is shorter than the oscillation period and therefore there should be no oscillations in observable quantities. In a general case the characteristic length $l_{tr}$ of the LRTC decay is $l_{tr} \approx \min\{1/Q, \xi_N\}$.

In the next section we discuss a possibility to observe the unusual behavior of the LRTC and demonstrate that such an observation is realistic.

## III. JOSEPHSON EFFECT IN *SFS* JUNCTION

In this section we consider a Josephson *SFS* junction with the same spiral magnetic structure as in the preceding sec-

tion. This junction is shown schematically in Fig. 1(b): A ferromagnetic layer with the spiral structure connects two superconductors where the phases of the order parameter are equal to $\pm\varphi/2$. We assume again a weak proximity effect and consider an important case where the Josephson coupling between the superconductors is due the LRTC only. This means that the condition

$$\kappa_h L \gg 1 \tag{28}$$

should be fulfilled. In this case the singlet component decays fast near the $S/F$ interfaces and the Josephson coupling is provided by an overlap of the LRTC.

In order to find the Josephson current $I_J$, we need to solve Eq. (12) with boundary conditions at $x = \pm L$. The boundary conditions coincide with Eq. (13), where in this case the matrix $\check{f}_S$ at $x = \pm L$ is replaced by

$$\check{f}_S = f_S(\hat{\tau}_2\cos(\varphi/2) \pm \hat{\tau}_1\sin(\varphi/2))\hat{\sigma}_3. \tag{29}$$

The matrix $\check{f}(x)$ is again represented in the form of an expansion in the Pauli matrices [see Eq. (14)], but the "coefficients" $\hat{F}(x)_i$ in this expansion acquire a somewhat more complicated form. These matrices (in the particle-hole space) have the structure (see Appendix B)

$$\hat{F}(x)_{0,3} = \hat{\tau}_2\sum_k A_{0,3k}\cosh(\kappa_k x) + \hat{\tau}_1\sum_k \mathcal{A}_{0,3k}\sinh(\kappa_k x). \tag{30}$$

The matrix $\hat{\tau}_3\hat{F}(x)_2$ has the same structure. As concerns the matrix $\hat{\tau}_3\hat{F}(x)_1$, it has a similar structure but with another spatial dependence

$$\hat{\tau}_3\hat{F}(x)_1 = \hat{\tau}_2\sum_k A_{1k}\sinh(\kappa_k x) + \hat{\tau}_1\sum_k \mathcal{A}_{1k}\cosh(\kappa_k x). \tag{31}$$

The coefficients $A_{1k}, \mathcal{A}_{1k}$ can be found from a solution for Eq. (12) in a way similar to that in the preceding section.

The Josephson current (per unit square) is calculated using a general formula for the condensate current in the dirty case (see, for example, Refs. 25 and 28)

$$I_J = i(\pi T/4\rho)\sum_\omega \text{Tr}\{\hat{\tau}_3 \cdot \hat{\sigma}_0 \check{f}\,\partial\check{f}/x\}, \tag{32}$$

where $\rho$ is the resistivity of the $F$ metal. Making the transformation (11), one can rewrite this formula in terms of the new function $\check{f}_n$

$$I_J = (i\pi T/4\rho)\sum_\omega Tr\{(\hat{\tau}_3 \cdot \hat{\sigma}_0)\check{f}(\partial\check{f}/x + i\hat{\tau}_3(Q/2)[\check{f}, \hat{\sigma}_1]_+)\}. \tag{33}$$

We dropped again the index "$n$." After simple but somewhat cumbersome calculations we obtain for $I_J$

$$I_J = I_c \sin \varphi, I_c$$

$$= \pi T(\sigma/2)|Q| \sum_{\omega,\alpha} \left\{ \mathcal{A}_{1\alpha} A_{1\alpha} \left(1 - \frac{(\zeta_\alpha - 1)^2}{4 \sin^2 \vartheta}\right) \bigg/ \sqrt{\zeta_\alpha} \right\}, \tag{34}$$

where the summation over $\alpha$ means that $\alpha=a$ and $\alpha=b$.

The amplitudes $A_{1a}, \mathcal{A}_{1a}$ can be found as before. We find (see Appendix B)

$$A_{1a} = \frac{2\zeta_a}{(\zeta_a + 1)\sinh \theta_a} \frac{i(A_{3-} - A_{3+})\cos \vartheta}{M}, \tag{35}$$

$$\mathcal{A}_{1a} = \frac{2\zeta_a}{(\zeta_a + 1)\cosh \theta_a} \frac{i(\mathcal{A}_{3-} - \mathcal{A}_{3+})\cos \vartheta}{\mathcal{M}}, \tag{36}$$

where $M = \sqrt{\zeta_a}/\tanh \theta_a - \sqrt{\zeta_b}/\tanh \theta_b$, $\mathcal{M} = \sqrt{\zeta_a}\tanh \theta_a - \sqrt{\zeta_b}\tanh \theta_b$, $\theta_{a,b} = \kappa_{a,b}L$. The coefficients $A_{1b}, \mathcal{A}_{1b}$ are given by the same formula with replacement $a \rightarrow b$. Under the condition (28) the coefficients for the singlet component $A_{3\pm}$ are described by formulas similar to Eq. (23)

$$A_{3\pm} = \frac{f_S}{2\gamma_B \kappa_\pm} \cos(\varphi/2); \quad \mathcal{A}_{3\pm} = \frac{f_S}{2\gamma_B \kappa_\pm} \sin(\varphi/2). \tag{37}$$

Equations (34)–(37) determine the Josephson critical for the *SFS* junction under consideration. In the approximation (18) we can perform the summation over $\omega$ and obtain for the critical current

$$I_c = \frac{2\pi|Q|}{\rho(\gamma_B \kappa_h)^2} \Delta \tanh\left(\frac{\Delta}{2T}\right) J_c. \tag{38}$$

Here the coefficient $\gamma_B \kappa_h$ is assumed to be large. Only in this case the Usadel equation may be linearized. However the obtained results are valid qualitatively in the case when this factor is of the order 1. The quantity $J_c$ depends only on the angle $\vartheta$ and the product $QL$. It is equal to

$$J_c = \frac{\cos^2 \vartheta}{M\mathcal{M}} \sum_\alpha \left\{ \left(\frac{\zeta_\alpha}{\zeta_\alpha + 1}\right)^2 \left(1 - \frac{(\zeta_\alpha - 1)^2}{4\sin^2 \vartheta}\right) \frac{1}{\sqrt{\zeta_\alpha}\sinh 2\theta_\alpha} \right\}. \tag{39}$$

Here the roots $\zeta_{a,b}$ are given by Eq. (20). Since at some angles the roots $\zeta_{\alpha,b}$ have an imaginary part, one can expect that the normalized critical current changes its dependence on $L$ with varying angle $\vartheta$. First we demonstrate this analytically considering a limiting case. We assume that the overlap of the condensate induced by each superconductor is weak. This means that $|\theta_\alpha| \gg 1$ and therefore $\sinh 2\theta_{\alpha,b} \approx (1/2)\exp(2\theta_{\alpha,b})$, $\tanh \theta_\alpha \approx 1$. If in addition the angle $\vartheta$ is close to $\pi/2$, i.e., $\cos \vartheta \ll 1$. In this limit we obtain

$$J_c \approx \sqrt{2} \cos \vartheta \exp(-\sqrt{2}QL \cos \vartheta)\sin(2QL). \tag{40}$$

Therefore the normalized critical current $J_c(L)$ as a function of $L$ undergoes many oscillations with the period $(\pi/Q)$ on the long decay length $(\sqrt{2}Q \cos \vartheta)^{-1}$. It turns to zero at $\cos \vartheta \rightarrow 0$, but the decay length becomes infinite. We remind that there is a lower limit on $\cos \vartheta$. It was assumed that
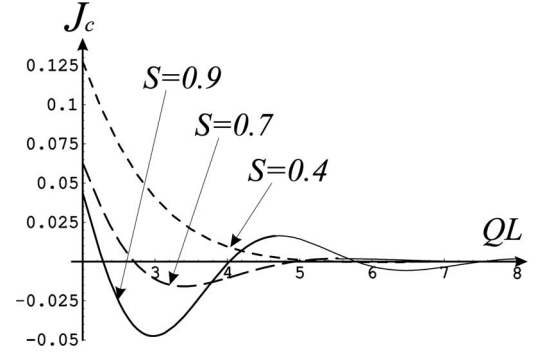


FIG. 3. Dependence of the normalized critical Josephson current $J_c$ on $QL$ for various $S = \sin \vartheta$.

$\cos \vartheta$ is larger than the small ratio $Q/\kappa_h$. The maximum of $J_c$ is achieved at $\cos \vartheta \approx 1/\sqrt{2}QL$ if $QL \gg 1$.

In Fig. 3 we plot the dependence of the critical current $I_c$ on the length $L$ for different projection of the magnetization on the $x$ axis. In accordance with the analysis above, it is seen that at the angle determined by $\sin \vartheta = 1/3$ the decay of $I_c$ with increasing $L$ is accompanied by oscillations. The period of these oscillations is of the order $Q^{-1}$ and depends on the angle $\vartheta$. These oscillations are caused by oscillations of the LRTC.

## IV. STRONG FERROMAGNET (QUASIBALLISTIC CASE)

In this section we consider the opposite limit, i.e., the condition

$$h\tau \gg 1, \tag{41}$$

is assumed to be fulfilled. This case may be realized either in a weak ferromagnet with a large mean free path or in a strong ferromagnet with a large exchange energy $h$. For example, this condition is fulfilled for $h \approx 1$ eV if the mean free path is longer than $\sim 50$ Å (we take the Fermi velocity $v = v_{F\uparrow} \approx v_{F\downarrow} \approx 10^8$ cm/s). For simplicity we assume that the **M** vector lies in the $\{y, z\}$ plane (that is, $\vartheta=0$). In order to find the condensate function in the ferromagnet, we have to use the more general Eilenberger equation. We assume again that the proximity effect is weak. The linearized Eilenberger equation for the condensate matrix function $\check{f}$ reads[29] (see Appendix A)

$$\mu l\{\hat{\tau}_3 \, \partial \check{f}/\partial x + i(Q/2)[\hat{\sigma}_1, \check{f}]_+\} + 2|\omega|\tau\check{f} - ih\tau \, \text{sgn} \, \omega[\hat{\sigma}_3, \check{f}]_+$$

$$= \langle \check{f} \rangle - \check{f}, \tag{42}$$

where $\mu = p_x/p$, $p_x$ is the projection of the momentum vector **p** on the $x$ axis, $l = \tau v$ is the mean free path, the angle brackets means the angle averaging. The boundary condition is[26]

$$(\text{sgn} \, \omega)\hat{\tau}_3 \check{a} = (\text{sgn} \, \mu)t_\mu \check{f}_{Sx=0}, \tag{43}$$

where $\check{a}(\mu) = [\check{f}(\mu) - \check{f}(-\mu)]/2$ is the antisymmetric part of the condensate function and $t_\mu = T(\mu)/4$, $T(\mu)$ is the transmission coefficient of the $S/F$ interface. In order to find the solutions of Eq. (42), we represent the matrix $\check{f}$ as a sum of antisymmetric and symmetric functions
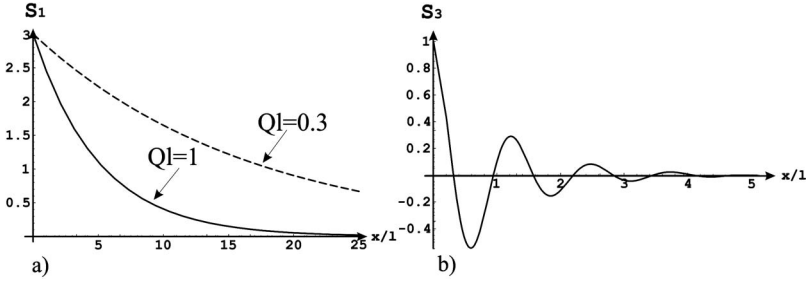
a)

b)

$$\check{f}(\mu) = \check{a}(\mu) + \check{s}(\mu), \qquad (44)$$

where $\check{s}(\mu) = [\check{f}(\mu) + \check{f}(-\mu)]/2$. The matrices $\check{a}(\mu)$ and $\check{s}(\mu)$ are represented again as a series in the spin matrices $\hat{\sigma}_i$

$$\check{a} = \sum_i \hat{a}_i \hat{\sigma}_i, \quad \check{s} = \sum_i \hat{s}_i \hat{\sigma}_i, \qquad (45)$$

where $i = 0, 1, 3$. The coefficients $\hat{a}_i, \hat{s}_i$ are, as before, matrices in the particle-hole space. We substitute these expansions into Eq. (42) and single out the symmetric and antisymmetric parts. After some algebra we obtain for the diagonal matrix elements in the spin space $(\check{s})_{11(22)} = \hat{s}_0 \pm \hat{s}_3 \equiv \hat{s}_\pm$ (see Appendix C)

$$\hat{s}_\pm(x) = \pm \hat{\tau}_2 t_\mu f_S \exp(-K_\pm x/l\mu), \qquad (46)$$

where $K_\pm = \alpha_\omega \mp i\alpha_h$, $\alpha_\omega = 1 + 2|\omega|\tau$, $\alpha_h = 2h\tau \operatorname{sgn} \omega$. The singlet $(\hat{s}_3)$ component and the triplet $(\hat{s}_0)$ component with zero projection of the magnetic moment of Cooper pairs are related to $\hat{s}_\pm(x)$: $\hat{s}_{3,0}(x) = [\check{s}_{11}(x) \mp \check{s}_{22}(x)]/2$. As is seen from Eq. (46), these components oscillate in space with a short period of the order of $v/h$ and decay over the mean free path $l$ (if $\tau T \ll 1$). When deriving the expression for $\hat{s}_{3,0}(x)$, we assumed that the condition

$$Ql/|\alpha_h| = Qv/h \ll 1 \qquad (47)$$

is satisfied, but the relation between the period $Q^{-1}$ of the spiral and the mean free path may be arbitrary.

Let us turn to the more interesting LRTC $\hat{s}_1$. An equation that describes the LRTC differs considerably from the one for the matrices $\hat{s}_\pm(x)$. Characteristic wave vectors are much smaller in this case. For the Fourier transform $\hat{s}_1(k) = \int dx \hat{s}_1(x) \exp(ikx)$ this equation has the form

$$[\alpha_\omega^2 + (Q^2(\alpha_\omega/\alpha_h)^2 + k^2)(\mu l)^2]\hat{s}_1(k) + (l\mu)^2(Qk)$$
$$\times \hat{\tau}_3[\hat{s}_0 + i(\alpha_\omega/\alpha_h)\hat{s}_3] = \alpha_\omega\langle \hat{s}_1 \rangle. \qquad (48)$$

The matrices $\hat{s}_{0,3}(x)$ are given in the first approximation by Eq. (46). Equation (48) can be solved in a general case, but we are interested in the behavior of the LRTC at distances much longer than the mean free path $l$. This means that one has to find the matrix $\hat{s}_1(k)$ for small $k$: $k \ll l$. We find (see Appendix C).

$$\hat{s}_1(k) = -(i\hat{\tau}_1)\frac{6Q}{k^2 + K_Q^2}\frac{\alpha_\omega^2}{\alpha_h}\langle \mu^2 t_\mu \rangle f_S, \qquad (49)$$

where $\langle \mu^2 t_\mu \rangle = \int_0^1 d\mu \mu^2 t_\mu$ and $K_Q^2 = 2|\omega|/D + (Q/\alpha_h)^2$. Performing the inverse Fourier transformation, we find the spatial dependence of the LRTC

$$\hat{s}_1(x) = -(i\hat{\tau}_1)3\frac{Q}{\alpha_h K_Q}\langle \mu^2 t_\mu \rangle f_S \exp(-xK_Q). \qquad (50)$$

We took into account that $\alpha_\omega = 1$. This is the main result of this section. The spatial dependence of the singlet component $\hat{s}_3(x) = (s_+ - s_-)/2$ is given by Eq. (46).

Comparing Eqs. (50) and (46), we see that the amplitude of the LRTC is comparable with the amplitude of the singlet component at the $S/F$ interface. Indeed, if $\sqrt{2\pi T/D} \ll Q/\alpha_h$, then $K_Q \approx Q/\alpha_h$ and the coefficient $(Q/\alpha_h K_Q)$ in Eq. (50) is of the order of 1.

In Fig. 4 we plot the spatial dependence of the LRTC and the singlet component. One can see that the singlet component oscillates fast with the period of the order $v/h = l/h\tau$ (we take the magnitude of $h\tau$ equal to 5) and decays over the mean free path $l$. The LRTC decays smoothly over a length $\sim (Q/h\tau)^{-1}$ [we assumed that $(Q/h\tau)^2 > 2\pi T/D$].

## V. CONCLUSIONS

We considered the odd triplet component of the superconducting condensate in $S/F$ systems with a spiral magnetic structure. The axis of the spiral is assumed to be perpendicular to the $S/F$ interface (the Bloch-like spiral structure). We analyzed both dirty ($h\tau < 1$) and clean ($h\tau > 1$) limits. These limits correspond in practice to the cases of weak and strong ferromagnets.

In the diffusive limit we studied the case of a conical ferromagnet when the magnetization vector $M$ has the constant projection $M \sin \vartheta$ on the $x$ axis and rotates around this axis. The condensate amplitude in the ferromagnet was assumed to be small compared to its amplitude in the superconductor (a weak proximity effect). In addition, we assumed that the exchange energy $h$ was larger than such energy scales as $T, DQ^2$ (dirty limit), $Qv$ (clean limit). In this case the penetration length of the singlet component is much less than that of the LRTC. The singlet component penetrates the ferromagnet F over a distance of the order $\sqrt{D/h}$ in the dirty limit and over the mean free path $l$ in the nearly clean limit (if $\tau T \ll 1$). In the ballistic case ($\tau T \gg 1$) the singlet component decays over distance of order $v/T$. In the nearly clean or ballistic case the singlet component oscillates fast

with the period $v/Q$. The LRTC decays over a length of the order of $\min\{Q^{-1}, \xi_N\}$.

In conical ferromagnets the LRTC has an interesting nontrivial dependence on the cone angle $\vartheta$. At small $\vartheta$ the LRTC decays exponentially in a monotonic way over the length $1/Q$ (if $Q \gtrsim \xi_N^{-1}$), but at $\vartheta \gtrsim \sin^{-1}(1/3) \approx 19°$ the exponential decay of the LRTC is accompanied by oscillations. The period of these oscillations depends on $\vartheta$ so that at $\vartheta \rightarrow \pi/2$ the period of oscillations is much smaller than the decay length. The amplitude of the LRTC is comparable with the amplitude of the singlet component at the $S/F$ interface. The latter amplitude is determined by the $S/F$ interface transmittance and decreases with increasing $h$.

In the dirty limit we calculated also the critical Josephson current $I_c$ for a $S/F/S$ junction with a conical ferromagnet $F$. It was assumed that the thickness of the $F$ layer $2L$ is much larger than $\xi_F$. Therefore the Josephson coupling is only due to an overlap of the LRTCs whereas the overlap of the singlet components induced by superconductors is negligible. The dependence of $I_c$ on the angle $\vartheta$ is determined by the LRTC: At small $\vartheta$ the critical current $I_c$ decreases with $L$ monotonously, but with increasing $\vartheta$ the decay of the function $I_c(L)$ is accompanied by oscillations. Therefore measurements of the Josephson critical current in $SFS$ junctions with a conical ferromagnet may provide useful information about the LRTC. Note that the triplet component may also exist in magnetic superconductors with a spiral magnetic structure (see Ref. 1 and references therein). However, in magnetic superconductors the triplet component coexists with the singlet one and, contrary to our case, cannot be separated from the singlet superconductivity.

We also studied the LRTC in the limit $h\tau > 1$ for $\vartheta = 0$. In this case the singlet component decays over a length of the order of the mean free path $l$ (if $\tau T \ll 1$). Its amplitude at the $S/F$ interface is determined by the $S/F$ interface transmittance and does not depend on $h$. The LRTC penetrates the ferromagnet over a length of the order of $(Q/h\tau)^{-1}$ [if $Q > (h\tau)/\xi_N$]. The decay length of the LRTC is longer than the decay length of the singlet component $l$ provided the condition $h > Qv$ is valid.

Note that we neglected the spin-orbit interaction. The latter restricts the penetration length of the LRTC by the value of the order of $D/8\tau_{so}$ where $\tau_{so}$ is the spin-orbit relaxation time.[25]

### ACKNOWLEDGMENT

### APPENDIX A: BASIC EQUATIONS; DIRTY CASE

The Eilenberger equation in a stationary case for the system under consideration has the form

$$\mu v \frac{\partial \check{g}}{\partial x} - i\epsilon [\hat{\tau}_3 \hat{\sigma}_0, \check{g}] - [\hat{\Delta} \hat{\sigma}_3, \check{g}] - ih\{\cos \vartheta([\hat{\tau}_3 \hat{\sigma}_3, \check{g}]\cos \alpha(x)$$

$$+ [\hat{\sigma}_2, \check{g}]\sin \alpha(x)) + \sin \vartheta[\hat{\sigma}_1, \check{g}]) + \frac{1}{2\tau}[\langle \check{g} \rangle, \check{g}] = 0,$$

$$(A1)$$

where $\mu = p_x/p$, $v$ is the Fermi velocity, $\check{g}$ is a $4 \times 4$ matrix of the retarded (advanced) Green's functions. The order parameter $\hat{\Delta}$ is not zero only in the superconductor and the exchange field $h$ differs from zero only in the ferromagnet. If the phase in the superconductor is chosen to be zero, then $\hat{\Delta} = i\hat{\tau}_2 \Delta$. The angle brackets mean the angle averaging. In the Matsubara representation the energy $\epsilon$ is replaced by $\omega$: $\epsilon \rightarrow i\omega$. If the condition $h\tau \ll 1$ is fulfilled, the antisymmetric in momentum **p** part of $\check{g}(\mathbf{p})$ is small and can be expressed through the symmetric part of $\check{g}$. For the symmetric part $\check{g}$ one can obtain the Usadel equation (see, for example, Ref. 25) that in the Matsubara representation reads

$$D \partial(\check{g} \partial \check{g})/\partial x^2 + \omega[\hat{\tau}_3 \hat{\sigma}_0, \check{g}] - [\hat{\Delta} \hat{\sigma}_3, \check{g}]$$

$$+ ih\{\cos \vartheta([\hat{\tau}_3 \hat{\sigma}_3, \check{g}]\cos \alpha(x) + [\hat{\sigma}_2, \check{g}]\sin \alpha(x))$$

$$+ \sin \vartheta[\hat{\sigma}_1, \check{g}]) = 0,$$

$$(A2)$$

where $D = vl/3$ is the diffusion constant.

In the case of a weak proximity effect one can linearize Eqs. (A1) and (A2). For example, in order to obtain the linearized Usadel equation for a small condensate function $\check{f}(x)$ in the ferromagnet, we represent $\check{g}(x)$ in the form

$$\check{g}(x) = \text{sgn } \omega \hat{\tau}_3 \hat{\sigma}_0 + \check{f}(x),$$

$$(A3)$$

where the first term is the matrix quasiclassical Green's function of a normal metal. Linearizing Eq. (A1) with respect to $\check{f}(x)$, we come to Eq. (8).

In order to find solutions for the matrix $\check{f}(x)$, we substitute the representation of this matrix $\check{f}(x)$ in the form of Eq. (14) and Eqs. (21) and (22) into Eq. (12). As a result we obtain on the left-hand side of this equation a sum of four terms proportional to the matrices $\hat{\sigma}_i$. Coefficients at each matrix $\hat{\sigma}_i$ are matrices in the particle-hole space. The sum of these four terms equals zero. Therefore we obtain four equations for these coefficients at $\hat{\sigma}_i$, where $i = 0, 1, 2, 3$

$$(-\kappa^2 + Q^2 + 2\kappa_\omega^2)\hat{F}_0(\kappa) + 2iQ\kappa[\hat{\tau}_3 \hat{F}_1(\kappa)]$$

$$- 2i \cos \vartheta \kappa_h^2 \hat{F}_3(\kappa) = 0,$$

$$(A4)$$

$$2iQ\kappa\hat{F}_0(\kappa) + (-\kappa^2 + Q^2 + 2\kappa_\omega^2)[\hat{\tau}_3 \hat{F}_1(\kappa)] = 0, \quad (A5)$$

$$(-\kappa^2 + 2\kappa_\omega^2)[\hat{\tau}_3 \hat{F}_2(\kappa)] - 2 \sin \vartheta \kappa_h^2 \hat{F}_3(\kappa) = 0, \quad (A6)$$

$$- 2i \cos \vartheta \kappa_h^2 \hat{F}_0(\kappa) + 2 \sin \vartheta \kappa_h^2 [\hat{\tau}_3 \hat{F}_2(\kappa)]$$

$$+ (-\kappa^2 + 2\kappa_\omega^2)\hat{F}_3(\kappa) = 0.$$

$$(A7)$$

This system of equations has a nonzero solution if the determinant of the system is zero. Thus we come to Eq. (15) for four eigenvalues $\kappa_k$. In order to determine the matrices $\hat{F}_i$, we have to use the boundary conditions (13). Since the matrix $\check{f}_S$ in the particle-hole space contains only the matrix $\hat{\tau}_2$, the matrices $\hat{F}_{0,3}(\kappa)$ and $\hat{\tau}_3 \hat{F}_{1,2}(\kappa)$ also are proportional to the matrix $\hat{\tau}_2$. Therefore one can write the expansion (21) and (22). The amplitude of each mode are $A_{i,k}$, where the first

index $i$ is related to the spin space and the second $k$ mean the eigenvalues of the wave vectors $\kappa_k (k = \pm, a, b)$. From Eqs. (A4)–(A7) one can determine relations between amplitudes $A_{i,k}(\kappa_k)$ for each mode. As follows from Eqs. (A4)–(A7), the coefficients $A_i(\kappa_k)$ are connected with each other via Eqs. (23), (24), and (24) provided the conditions (16) and (18) are fulfilled. The matrices $[\hat{\tau}_3 \hat{F}_1(\kappa_\pm)] \approx (2iQ/\kappa_\pm)\hat{F}_0(\kappa_\pm)$ and $\hat{F}_3(\kappa_{a,b}) \approx -(Q/\kappa_h)^2 (z_{a,b}/\sin \vartheta)^2 \hat{F}_0(\kappa_\pm)$ are small compared to other matrices. Using these relations and substituting the representation (14) into the boundary condition (13), we obtain four equations for the amplitudes $A_{i,k}(\kappa_k)$ $(i = 0, 1, 2, 3)$

$$\sum_{n,\alpha} \{\kappa_n A_{0n} + \kappa_\alpha A_{0\alpha} - iQ A_{1\alpha}\} = 0, \qquad (A8)$$

$$\sum_{n,\alpha} \{\kappa_n A_{1n} + \kappa_\alpha A_{1\alpha} - iQ(A_{0n} + A_{0\alpha})\} = 0, \qquad (A9)$$

$$\sum_{n,\alpha} \{\kappa_n A_{2n} + \kappa_\alpha A_{2\alpha}\} = 0, \qquad (A10)$$

$$\sum_{n,\alpha} \{\kappa_n A_{3n} + \kappa_\alpha A_{3\alpha}\} = (f_S/\gamma_b), \qquad (A11)$$

where the summation is performed over the eigenvalues $n = \pm, \alpha = a, b$; that is, the indices $n$ and $\alpha$ correspond to the short- and long-range eigenfunctions, respectively. Solutions for these equations yield Eqs. (23)–(25).

## APPENDIX B: JOSEPHSON EFFECT

We consider the limit $\kappa_h L \gg 1$ when one can neglect the overlap of the short-range eigenfunctions corresponding to the eigenvalues $\kappa_\pm$. In this case solutions for $\hat{F}_{0,1}(x)$ may be represented in the form

$$\hat{F}_0(x) = \sum_{n,\alpha} (A_{0n}\hat{\tau}_2 \pm \mathcal{A}_{0n}\hat{\tau}_1)\exp[-\kappa_n(L \mp x)]$$

$$+ A_{0\alpha}\hat{\tau}_2 \cosh(\kappa_\alpha x) + \mathcal{A}_{0\alpha}\hat{\tau}_1 \sinh(\kappa_\alpha x), \quad (B1)$$

where $n, \alpha$, as before, are equal to $\pm$ and $a, b$. The matrix $\hat{F}_2(x)$ has a similar form. The amplitude of the long-range component of $\hat{F}_3(x)$ is small. This can be seen from Eq. (A4). The matrix $\hat{F}_1(x)$ has a different spatial dependence

$$\hat{F}_1(x) = \hat{\tau}_3 \sum_{n,\alpha} (A_{1n}\hat{\tau}_2 \pm \mathcal{A}_{1n}\hat{\tau}_1)\exp[-\kappa_n(L \mp x)]$$

$$+ A_{1\alpha}\hat{\tau}_2 \cosh(\kappa_\alpha x) + \mathcal{A}_{1\alpha}\hat{\tau}_1 \sinh(\kappa_\alpha x). \quad (B2)$$

The first term in Eqs. (B1) and (B2) describes the modes fast decaying from the $S/F$ interfaces at $x = \pm L$ and the second term corresponds to the LRTC. The coefficients $A_{1n}, A_{1\alpha}$ and $\mathcal{A}_{0n}, \mathcal{A}_{0\alpha}$ are connected with each other by Eqs. (A4)–(A7) [see Eqs. (23) and (24)]. The additional terms $\mathcal{A}_{0,1n,\alpha}\hat{\tau}_1$ appear because the matrix $\check{f}_S$ has changed [see Eq. (29)]. In order to find these amplitudes, one has to substitute the expressions Eqs. (B1) and (B2) into the boundary conditions

$$\partial \check{f}/\partial x + i(Q/2)\hat{\tau}_3[\hat{\sigma}_1, \check{f}]_{+x=\pm L} = (f_S/\gamma_b)[\hat{\tau}_2 \cos(\varphi/2)$$

$$\pm \hat{\tau}_1 \sin(\varphi/2)] \cdot \hat{\sigma}_3. \quad (B3)$$

Performing the calculations in this way, we arrive at four equations for the matrices $\hat{F}_i(L)$ [compare with Eqs. (A8)–(A11)]

$$\sum_{n,\alpha} \{\kappa_n A_{0n} + \kappa_\alpha A_{0\alpha}\sinh \theta_\alpha + iQA_{1\alpha}\sinh \theta_\alpha\} = 0, \quad (B4)$$

$$\sum_{n,\alpha} \{(\kappa_n A_{1n} + \kappa_\alpha A_{1\alpha}\cosh \theta_\alpha) + iQ(A_{0n} + A_{0\alpha}\cosh \theta_\alpha)\} = 0, \quad (B5)$$

$$\sum_{n,\alpha} \{-i \tan \vartheta \hat{\tau}_3 \kappa_n A_{0n} + i \cot \vartheta \kappa_\alpha A_{0\alpha}\sinh \theta_\alpha\} = 0, \quad (B6)$$

$$\sum_{n,\alpha} \{\kappa_n A_{3n} + \kappa_\alpha A_{3\alpha}\cosh \theta_\alpha\} = (f_S/\gamma_b)\cos(\varphi/2). \quad (B7)$$

We expressed $\hat{F}_{2n,\alpha}$ in terms of $\hat{F}_{0n,\alpha}$ making use Eqs. (23) and (24). The corresponding equations for the coefficients $\mathcal{A}_{i,n,\alpha}$ may be obtained in a similar way. These equations coincide with Eqs. (B4)–(B7) if one makes the replacement $\sinh \theta_\alpha \rightleftarrows \cosh \theta_\alpha$ and $\cos(\varphi/2) \rightarrow \sin(\varphi/2)$. In the main approximation in the parameter $Q/\kappa_h$ solutions for these equations are given by Eqs. (35)–(37).

## APPENDIX C: QUASIBALLISTIC CASE

In this section we represent formulas for the condensate function $\check{f}(x)$ in the case of a strong ferromagnet or a large mean free path $l$ when the condition (41) is fulfilled. Substituting Eq. (A3) into Eq. (A1) and performing the transformation (11), we obtain the linearized Eilenberger equation for the new function $\check{f}_n(x)$ in the ferromagnet (for brevity we drop the subindex $n$)

$$\mu \hat{\tau}_3 l \, \partial \check{f}/\partial x + i(Q/2)\mu l[\hat{\sigma}_1, \check{f}]_+ + \alpha_\omega \check{f} - i(\alpha_h/2)[\hat{\sigma}_3, \check{f}]_+ = \langle \check{f} \rangle, \quad (C1)$$

where $\alpha_\omega = 1 + 2|\omega|\tau, \alpha_h = 2h\tau \operatorname{sgn} \omega$. In order to solve this equation, we represent the matrix $\check{f}(x)$ as a sum of matrices symmetric $\check{s}(x)$ and antisymmetric $\check{a}(x)$ in the momentum space

$$\check{f}(x) = \check{s}(x) + \check{a}(x). \quad (C2)$$

Equations for these matrices can be obtained if we substitute Eq. (C2) into Eq. (C1) and split it into the symmetric and antisymmetric in $\mu$ parts. We write down, for example, equations for the diagonal elements of matrices $\check{s}(x)$ and $\check{a}(x)$ in the spin space: $\check{s}(x)_{11(22)} = \hat{s}_\pm, \check{a}(x)_{11(22)} = \hat{a}_\pm$. These equation have the form

$$(K_\pm^2 - \mu^2 l^2 \partial^2/\partial x^2)\hat{s}_\pm + (Ql\mu)^2 (K_\pm/\alpha_\omega)\hat{s}_0$$

$$= i(\mu l)^2 Q[1 + K_\pm/\alpha_\omega]\hat{\tau}_3 \, \partial \hat{s}_1/\partial x + K_\pm \langle \hat{s}_\pm \rangle, \quad (C3)$$

$$K_\pm \hat{a}_\pm = -(\mu l)[\hat{\tau}_3 \partial_x \hat{s}_\pm + iQ\hat{s}_1]. \tag{C4}$$

The coefficients $K_\pm$ are defined in Eq. (46). We solve equations for $\hat{s}_\pm$ assuming that the coefficients $K_\pm$ are large, that is, $\alpha_h$ is large. In addition, we assumed that the ratio $Ql/\alpha_h = Qv/h$ is small. In this case the second term on the left-hand side and all terms on the right-hand side of Eq. (C3) can be neglected. Solving this equation with the boundary conditions (43), in the main approximation we obtain the expression (46). One can easily check that the term $\langle \hat{s}_\pm \rangle$ is much smaller than $K_\pm \hat{s}_\pm$ provided the quantity $\alpha_h$ is large. The equation for the matrix $\hat{s}_1$ can be readily obtained in a similar way. We get

$$\left[ \alpha_\omega^2 + (\mu l)^2 Q^2 \left( \frac{\alpha_\omega}{\alpha_h} \right)^2 - \mu^2 l^2 \partial^2/\partial x^2 \right] (\hat{\tau}_3 \hat{s}_1) - (l\mu)^2 iQ \partial_x$$

$$\times \left[ \hat{s}_0 + i\frac{\alpha_\omega}{\alpha_h} \hat{s}_3 \right] = \alpha_\omega \langle \hat{\tau}_3 \hat{s}_1 \rangle. \tag{C5}$$

The boundary condition for $\hat{\tau}_3 \hat{s}_1(x)$ at $x=0$ requires that $\hat{a}_1 = 0$ at $x=0$. The expression for the antisymmetric matrix $\hat{a}_1$ has the form

$$\alpha_\omega \hat{a}_1 = -(l\mu)[\partial(\hat{\tau}_3 \hat{s}_1)/\partial x + iQ\hat{s}_0]. \tag{C6}$$

As follows from Eq. (46), at the $S/F$ interface in the main approximation $\hat{s}_0(0)=0$. Therefore the boundary condition for the matrix $\hat{\tau}_3 \hat{s}_1(x)$ may be written as

$$\partial_x \hat{s}_1 = 0. \tag{C7}$$

Equation (C5) can be solved in the following way. In the main approximation the coordinate dependence of the matrices $\hat{s}_{0,3}(x)$ is given by Eq. (46) $[\hat{s}_{0,3} = (\hat{s}_+ \pm \hat{s}_-)/2]$ and these functions vary over distances $|l/K_\pm| \approx v/h$ that are much shorter than a characteristic scale for the LRTC variation. Therefore approximately we can represent a solution for Eq. (C5) in the form

$$\partial_x \hat{s}_1(x) = -\hat{\tau}_3 iQ[\hat{s}_0(x) + i(\alpha_\omega/\alpha_h)\hat{s}_3(x)] + \partial_x \delta \hat{s}_1(x). \tag{C8}$$

In particular the matrix $\delta \hat{s}_1(x)$ contains the LRTC. As follows from Eqs. (C6) and (C7), the boundary condition for the function $\delta \hat{s}_1(x)$ is

$$\partial_x \delta \hat{s}_1(x) = -Q[(\alpha_\omega/\alpha_h)\hat{s}_3(x)]_{x=0}, \tag{C9}$$

where $\hat{s}_3(0)=\hat{\tau}_2 t_\mu f_S$. The equation for $\delta \hat{s}_1(k)$ in the Fourier representation $[\delta \hat{s}_1(k) = \int dx \, \delta \hat{s}_1(x) \exp(ikx)]$ can be easily obtained from Eqs. (C5), (C8), and (C9). It has the form

$$\left[ \alpha_\omega^2 + Q^2 \left( \frac{\alpha_\omega}{\alpha_h} \right)^2 + (k\mu l)^2 \right] \delta \hat{s}_1(k)$$

$$= \alpha_\omega \langle \delta \hat{s}_1(k) \rangle + 2(l\mu)^2 Q \frac{\alpha_\omega}{\alpha_h} \hat{\tau}_3 \hat{s}_3(0). \tag{C10}$$

From this equation one can easily find $\delta \hat{s}_1(k)$

$$\delta \hat{s}_1(k) = \hat{\tau}_3 \frac{2Ql^2}{N(k,\mu)} \frac{\alpha_\omega}{\alpha_h} \left\{ \frac{\alpha_\omega}{N_{\mathrm{LR}}(k)} \left\langle \frac{\mu^2}{N(k,\mu)} \hat{s}_3(0) \right\rangle + \mu^2 \hat{s}_3(0) \right\}, \tag{C11}$$

where $N(k,\mu)=\alpha_\omega^2 + Q^2(\alpha_\omega/\alpha_h)^2 + (k\mu l)^2$ and $N_{\mathrm{LR}}(k) = 1 - \alpha_\omega \langle 1/N(k,\mu) \rangle$. The behavior of the LRTC $\hat{s}_1(x)$ is determined by poles of the functions $N(k,\mu)$ and $N_{\mathrm{LR}}(k)$. The first function has poles at $k \approx l^{-1}$. These poles determine a variation of the matrix $\hat{s}_1(x)$ over distances of the order of the mean free path from the S/F interface. The function $N_{LR}(k)$ has poles at much smaller wave vectors $k$ which determine a long-range penetration of the triplet component. Indeed for $k \ll l^{-1}$ we have: $N_{\mathrm{LR}}(k) = (l^2/3)[k^2 + K_Q^2]$ with $K_Q^2 = 2|\omega|\tau/D + (Q/\alpha_h)^2$. Therefore at these wave vectors $k$ the expression for $\delta \hat{s}_1(k)$ is reduced to Eq. (48).

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